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PROBLEMS IN ANALYSIS

Introduction

This is a collection of problems from when I taught Calc 114 at University of Pennsylvania. Questions and suggestions are welcome at `per.w.alexandersson@gmail.com`.

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Basic integration techniques

Problem. 1
Compute the following integrals by making an appropriate substitution:

$$a) \int e^{\sqrt{x}} dx \qquad b) \int \frac{x^3}{1+x^8} dx$$

Solution. 1

Problem. 2
Compute

$$\int \sqrt{1+t^2} dt$$

using the substitution $t = \tan u$ or $t = \sinh u$.

Solution. 2
Using $t = \tan u$ gives $dt = du/\cos^2(u)$. The integral turns into

$$\int \frac{\sqrt{1+\tan^2 u}}{\cos^2 u} du.$$

The next step is to see the following:

$$1 + \tan^2 u = \frac{\cos^2 u}{\cos^2 u} + \frac{\sin^2 u}{\cos^2 u} = \frac{\cos^2 u + \sin^2 u}{\cos^2 u} = \frac{1}{\cos^2 u}$$

The integral then simplifies to

$$\int \frac{1}{\cos^3 u} du = \int \frac{1}{\cos^2 u} \frac{1}{\cos u} du$$

From here, we do integration by parts.

Problem. 3
Compute

$$\int \sin(t) e^t dt$$

using integration by parts.

Selected problems

The Jump

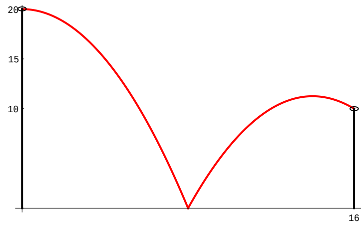


Figure 1: The planned path.

Problem. 4

You live 20m above ground and have a window facing a yard, which is 16m wide.

Your neighbor has an open window 10m from the ground, also facing the yard. There is an industrial-strength trampoline that retains 75% of the vertical speed in the yard between your houses. You start running horizontally. Can you make it?

Gravity is assumed to be $10m/s^2$.

Solution. 4

We set up the model as usual, with $(s_x, 0)$ as initial velocity and $(0, 20)$ as initial position. The path before the bounce, starting at $t = 0$, is then given by

$$\mathbf{p}(t) = (0, 20) + (s_x, 0)t + (0, -5)t^2.$$

We seek the time of impact, meaning we seek t such that the y-coordinate vanish:

$$20 - 5t^2 = 0 \qquad \Longrightarrow \qquad t = 2.$$

Thus, we hit the trampoline after two seconds, independent of the initial speed. The position of impact is given by $\mathbf{p}(2) = (2s_x, 0)$. We now compute the velocity by taking the derivative of the position function:

$$\mathbf{p}'(t) = (s_x, 0) + (0, -10)t.$$

Velocity at impact ($t = 2$) is then $\mathbf{p}'(2) = (s_x, -20)$. Bouncing retains $3/4$ of the vertical speed and preserves the horizontal speed. Therefore, the velocity immediately *after* the bounce is given by $(s_x, \frac{3}{4} \cdot 20) = (s_x, 15)$, since the vertical direction is reflected.

At the point of first impact, the vertical speed is about $70km/h$. If you did not find a correct solution, the following reference might be useful: H. D. Haven, *Mechanical analysis of survival in falls from heights of fifty to one hundred and fifty feet*, Injury prevention 2000;**6**:62–68. http://injuryprevention.bmj.com/content/6/1/62.3.long

We now model the second path, $\mathbf{q}(t)$, where $t = 0$ represents seconds after the bounce. Note that at $t = 0$, we are at the position of impact, $(2s_x, 0)$. This gives

$$\mathbf{q}(t) = (2s_x, 0) + (s_x, 15)t + (0, -5)t^2.$$

We seek s_x and t such that $\mathbf{q}(t) = (16, 10)$. In other words, we have the equation

$$(2s_x, 0) + (s_x, 15)t + (0, -5)t^2 = (16, 10).$$

Decomposing into components gives

$$\begin{cases} 2s_x + s_x t &= 16 \\ 15t - 5t^2 &= 10. \end{cases}$$

The second equation gives $t = 1$ and $t = 2$ as solutions.

- **Case** $t = 1$**:** This gives $2s_x + s_x = 16$, and $s_x = 16/3$ meters per second.
- **Case** $t = 2$**:** This gives $2s_x + 2s_x = 16$, and $s_x = 4$ meters per second.

It follows that we succeed if we run out the window with either 4 or 16/3 meters per second, with the paths illustrated in the picture.

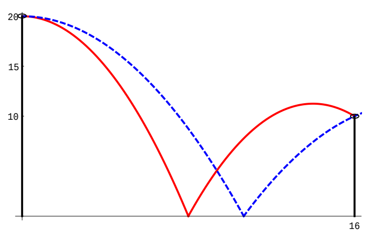


Figure 2: The two possible paths.

The London Eye incident

Problem. 5

There is a ferris wheel in London, called London Eye. It has diameter 120 meters and one revolution takes 30 min, if everything is normal.

In one particular day, the speed of the wheel starts to accelerate, such that the total number of revolutions after t *seconds* is given by $s(t) = t^2/10^4$.

We started at the bottom of the wheel at $t = 0$. Once the speed hits 360km/h the gondola breaks free and makes a spectacular journey into the river Thames. Assume that the river is at ground level.

Close to the record speed of a Formula 1 car

1. Describe the position of person staring at the bottom at $t = 0$, under normal operation of the London Eye (moving counter-clockwise).
2. If the incident start at $t = 0$, after how many minutes is the speed 360km/h? (Use calculator).
3. Determine the position and velocity of the gondola at this time.
4. How far from the London Eye does the gondola land?

Solution. 5

The position of the gondola is given by

$$r(t) = 120\sin(2\pi\frac{t}{2})i + 120(1+\cos(2\pi\frac{t}{2}))j$$

with t being total number of hours.

In general, the position of the gondola is $r(s(t))$ with

$$r(t) = 120\sin(2\pi s(t))i + 120(1+\cos(2\pi s(t)))$$

after $s(t)$ revolutions. The velocity at time t is now

$$r'(s(t))s'(t) = 120\cdot 2\pi[\cos(2\pi s(t))i - \sin(2\pi s(t))j]\cdot 2t\cdot 10^{-4}. \quad (1)$$

The speed is the magnitude of this vector: $48\cdot 10^{-3}\pi t$ meter per second.

We now solve

$$48\cdot 10^{-3}\pi t = 360/3.6$$

which gives $t \approx 663$ seconds, or 11 minutes. We plug this into $r(s(t))$ and get $r(s(663)) = (-32, 236)$, which is the position, let us call it p_0. The velocity is given by (1), which simplifies to $(96, 27)$ meter per second.

Once the gondola breaks free, its position is determined by

$$p(t) = (-32, 236) + (96, 27)t - 9.82\frac{t^2}{2}(0, 1)$$

The y-coordinate is $236 + 27t - 9.82t^2/2$ and we find that this is 0 for $t \approx 10$. For $t = 10$, the x-coordinate is about 930. In other words, the gondola landed almost one kilometer from the London Eye.

Professor A and the alternative scoring scheme

Problem. 6

In the hypothetical country of Swedistan, Prof. A teaches a multivariate calculus class with three midterms, each with a maximum score of 100. Prof. A allows each student to pick the weight each of midterm, such that the final score is given by $M_1 w_1 + M_2 w_2 + M_3 w_3$, where the w_i are non-negative weights. There are of course some conditions of the weight, and Prof. A contemplates on the two following alternatives.

Alternative 1: $w_1 + w_2 + w_3 = 1$.

Alternative 2: $w_1^2 + w_2^2 + w_3^2 = 1$.

Suppose Eva–Lina got the midterm scores 89, 58 and 62.

1. What weights should she pick to maximize the final score in the first alternative?
2. What about the second alternative? Use Lagrange multipliers!
3. The second alternative can be interpreted as maximizing the scalar product $M \cdot \mathbf{w}$, given that $|\mathbf{w}| = 1$, and $M = (M_1, M_2, M_3)$. What weights should a student with scores M pick?
4. Suppose Alice, Bob and Charlie got the scores $(90, 5, 5)$, $(60, 60, 60)$, and $(80, 40, 60)$, respectively. What is the final score for these students in the two grading schemes, and how does it compare with simply taking the average?

Method	Alice	Bob	Charlie
Alt. 1			
Alt. 2			
Avg. score			

Table 1: Final score

5. Which of the three grading scheme is the most fair? Discuss!
6. Why does Prof. A insist on students picking non-negative weights? Find a strategy in the first scoring alternative where a student might benefit from choosing a negative weight.

Optimization on compact regions

Find and classify *all* critical points of the functions, even the ones outside the region. Then proceed and find global minimum and maximum on the specified region.

Note: The answers only specify one point where the global min/max is obtained — there might be several others!

Problem. 7

Function $2x - y + x^2 + y^2$ in $x^2 + y^2 \leq 4$.

Solution. 7
Local min in $(-1, \frac{1}{2})$. Global min: $-\frac{5}{4}$ in $(-1, \frac{1}{2})$, global max: $2(2+\sqrt{5})$ in $(4/\sqrt{5}, -2/\sqrt{5})$.

Problem. 8
Function $\ln(x^2+y^2) - x - y$ in $\frac{1}{2} \leq x^2 + y^2 \leq 8$.

Solution. 8
Saddle in $(1, 1)$. Global min: $-1 - \ln(2)$ in $(\frac{1}{2}, \frac{1}{2})$, global max: $4 + \ln(8)$ in $(-2, -2)$.

Problem. 9
Function $xy^2 - x^2 - y^2$ in $x^2 + y^2 \leq 12$.

Solution. 9
Saddle in $(1, \sqrt{2})$ and $(1, -\sqrt{2})$, local max in $(0, 0)$. Global min: -28 in $(-2, 2\sqrt{2})$, global max: 4 in $(2, 2\sqrt{2})$.

Problem. 10
Function $xy^2 - 2x^2y - 3x$ in the triangle $x + y \leq 3$ and $x, y \geq 0$.

Answer: saddle in $(1, 1)$ and $(-1, -1)$. Global min: 0 in $(0, \frac{3}{2})$, global max: 9 in $(3, 0)$.

Problem. 11
Function $x^2 + xy + y^2$ in the unit disk.

Answer: local minimum in $(0, 0)$. Global min: 0 in $(0, 0)$, global max: $3/2$ in $(-\sqrt{2}/2, -\sqrt{2}/2)$.

Problem. 12
Function $x^2 - 2xy + 2y^2 - 2y$ in the triangle $2x + y \leq 4$ and $x, y \geq 0$.

Answer: local min in $(1, 1)$. Global min: -1 in $(1, 1)$, global max: 24 in $(0, 4)$.

Problem. 13
Function $\frac{x+y^2}{1+x^2+y^2}$ in $x^2 + y^2 \leq 4$.

Answer: local min in $(-1, 0)$, saddle in $(1, 0)$. Global min: $-\frac{1}{2}$ in $(-1, 0)$, global max: $17/20$ in $(1/2, -\sqrt{15}/2)$.

Problem. 14
Function $2xy^3 - x^2 - 3y^2$ in the square $0 \leq x, y \leq 2$.

Answer: local max in $(0, 0)$, saddle in $(-1, -1)$ and $(1, 1)$. Global min: -12 in $(0, 2)$, global max: 16 in $(2, 2)$.

Problem. 15
Function $xy(1 - x - y)$ in the triangle with corners $(0, 0)$, $(1, 0)$ and $(0, 1)$.

Answer: local max in $(1/3, 1/3)$, saddle in $(0, 0)$, $(0, 1)$, $(1, 0)$. Global min: 0 in $(1/2, 0)$, global max: $1/27$ in $(1/3, 1/3)$.

Problem. 16

Function $x^3 - 4xy + 2y^2$ in the triangle with corners $(0,0)$, $(2,4)$ and $(2,1)$. *Hint:* Show that the boundaries of the triangle are given by $x = 2$, $y = x/2$ and $y = 2x$.

Answer: local max in $(-3/2, 1/2)$, saddle in $(1/2, 1/2)$. Global min: -2 in $(0,1)$, global max: 0 in $(0,0)$.

Problem. 17

Function $x\frac{x^2+y^2}{1-x^2-y^2}$ in $x^2 + y^2 \leq 1/4$.

Answer: inconclusive point in $(0,0)$, saddle in $(\sqrt{3}, 0)$ and $(-\sqrt{3}, 0)$. Global min: $-\frac{1}{6}$ in $(-1/2, 0)$, global max: $\frac{1}{6}$ in $(1/2, 0)$.

Note: Near $(0,0)$, $f(x,0) \approx x^3$ and $f(0,y) = 0$, so the behavior is similar to an inflection point along the x-axis, and flat along the y-axis. We note that $f(x,0) < 0$ if $-1 < x < 0$ and $f(x,0) > 0$ if $0 < x < 1$, so $(0,0)$ has to be a saddle point — we can find values near $(0,0)$ that are both larger and smaller than the value at $(0,0)$.

The Jacobian

Suppose we want to do a change of variables in an integral. We express the *old* variable x, in terms of the *new* variable u, and dx is replaced by $g'(u)du$:

$$\int f(x)dx \qquad \Rightarrow \qquad \begin{cases} x = g(u) \\ dx = g'(u)du \end{cases} \qquad \Rightarrow \qquad \int f(g(u))g'(u)du$$

Example 1. Compute

$$\int \frac{dx}{\sqrt{1-x^2}}.$$

We express the old variable in terms of the new variable, $x = \sin(t)$, $dx = \cos(t)dt$, and we get

$$\int \frac{\cos(t)dt}{\sqrt{1-\sin^2(t)}}$$

which then can be further simplified.

However, in may instances, it is more convenient to express the *new* variable in the *old* variable.

$$\int f(x)dx \qquad \Rightarrow \qquad \begin{cases} u = h(x) \\ du = h'(x)dx \end{cases} \qquad \Rightarrow \qquad \int f(x)\frac{du}{h'(x)}$$

and finally, in the last expression, we replace x with $h^{-1}(u)$. The way we presented this is perhaps a bit abstract, but in many instances we can simplify $f(x)/h'(x)$ before doing the substitution, which is quite nice.

Example 2. Compute

$$\int e^{\sqrt{x}} dx$$

We express the new variable in terms of the old variable, $u = \sqrt{x}$, $du = \frac{dx}{2\sqrt{x}}$. Thus, $dx = 2\sqrt{x} du$ and we get

$$\int e^{\sqrt{x}} \cdot 2\sqrt{x} du = \int 2u \cdot e^u du.$$

Several variables

When we do a change of variables in *several* variables, the *Jacobian* takes the role of the inner derivative. That is, suppose we have *old* variables x and y, expressed in *new* variables as $x = g(u, v)$, $y = h(u, v)$. Then we get

$$\iint_E f(x, y) dx dy = \iint_{E'} f(g(u, v), h(u, v)) \cdot |J(u, v)| du dv$$

where

$$J(u, v) = \begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{vmatrix}$$

and we need to change the region E to E'.

Example 3. Compute

$$\iint_E e^{x^2+y^2} dx dy$$

where E is the unit circle. We use polar coordinates, and express the old coordinates in terms of the new coordinates,

$$x = r\cos(t) \qquad y = r\sin(t).$$

The Jacobian is then given by

$$J(r, t) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} \cos(t) & -r\sin(t) \\ \sin(t) & r\cos(t) \end{vmatrix} = r\cos^2 t + r\sin^2 t = r$$

and we get that

$$\iint_E e^{x^2+y^2} dx dy = \iint_{E'} e^{r^2} \cdot |r| \cdot dr dt.$$

Since $r > 0$, we can drop the absolute value, and E' is the *rectangular* region $0 \le r \le 1$, $0 \le t \le 2\pi$.

As in the one-variable case, it is sometimes much easier to express the *new* variables in the *old* variables. That is, we have $u = g(x, y)$, $v = h(x, y)$. Then we get

$$\iint_E f(x, y) dx dy = \iint_{E'} f(x, y) \cdot \frac{1}{|J(x, y)|} du dv$$

where

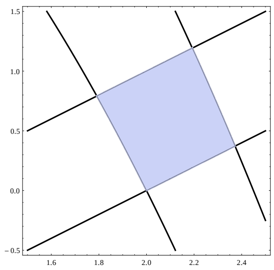
$$J(x,y) = \begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix}.$$

As before, we need to rewrite $f(x,y) \cdot \frac{1}{|J(x,y)|}$ in terms of u and v, but in some particular cases, this can be done *without* solving for x and y in terms of u and v.

Example 4. Compute the integral

$$\iint_E (1+2x)(x-y)e^{x^2+y} dxdy$$

over the region given by $4 \leq x^2 + y \leq 6$ and $1 \leq x - y \leq 2$.



Looking at the region, it is natural to pick $u = x^2 + y$ and $v = x - y$, since the new region E' would then be the rectangle $4 \leq u \leq 6$, $1 \leq v \leq 2$.

With this choice of change of coordinates, we get

$$J(x,y) = \begin{vmatrix} 2x & 1 \\ 1 & -1 \end{vmatrix} = -(2x+1).$$

Since $x > 0$ everywhere in the region E, we have that $|J(x,y)| = 2x + 1$ in E. Hence,

$$\iint_E (1+2x)(x-y)e^{x^2+y} dxdy = \iint_{E'} (1+2x)(x-y)e^{x^2+y} \frac{1}{(2x+1)} dudv.$$

Canceling and recognizing $v = x - y$ and $u = x^2 + y$, we get

$$\iint_{E'} (x-y)e^{x^2+y} dudv = \iint_{E'} ve^u dudv = \int_1^2 \int_4^6 ve^u dudv$$

Note that in the previous example, we never had to explicitly express x and y in terms of u and v. If we did, we would get

$$x = \frac{-1+\sqrt{1+4u+4v}}{2} \qquad y = \frac{-1-2v+\sqrt{1+4u+4v}}{2},$$

which is quite atrocious, and we need to do some careful argument[1] to determine which sign to put before the square roots.

In reality, we need to be careful and argue why the map $(x,y) \to (u,v)$ from E to $E's$ is invertible. One condition that needs to be fulfilled is that the Jacobian is never 0 in the region — and this is clearly true since $2x + 1 > 0$ in E. By explicitly computing the

[1] The choice of sign follows from that we know $x > 0$.

inverse, we can be certain that the map is invertible, but as we saw above, the result might look ugly.

In one special case you never have to check for the map to be invertible, namely linear maps, which are of the form $u = ax + by$, $v = cx + dy$ for constants a, b, c, d. Such maps are always invertible, as long as the Jacobian is not 0.

Integrals over regions, polar coordinates and Jacobians

Problem. 18
Compute

$$\int_1^2 \int_1^{y^2} \frac{y^3}{\sqrt{x}} dxdy + \int_2^4 \int_{y^2/4}^4 \frac{y^3}{\sqrt{x}} dxdy$$

by drawing the regions and then changing the order of integration.
Answer: 93/2.

Problem. 19
Compute the integral

$$\int_0^8 \int_{\frac{y^{2/3}}{4}}^1 5e^{x^{5/2}} dxdy.$$

Answer: $16(e-1)$.

Problem. 20
Compute

$$\int_{1/2}^1 \int_{\pi/y}^{2\pi} \sin(xy) dxdy + \int_1^2 \int_{\pi}^{2\pi/y} \sin(xy) dxdy$$

Hint: Draw the regions.
Answer: $-\ln(4)$.

Problem. 21
(Polar) Evaluate $\iint_R y dA$ where R is the upper half of the unit circle.

Answer: $2\pi/3$

Problem. 22
Find the average value of $f(x, y) = x + y$ on $0 \leq x, y \leq 8$. This is darts-on-a-chessboard problem.
Answer: 8

Problem. 23
(Polar) Integrate x^2 in the region $x \geq \frac{1}{2}$, $x^2 + y^2 \leq 1$.
Answer: $\frac{\pi}{12} + \frac{\sqrt{3}}{32}$.

Problem. 24
(Polar) Find $A = \int_{-\infty}^{\infty} e^{-x^2} dx$ by first noticing that

$$A^2 = \iint_{\mathbb{R}^2} e^{-x^2-y^2} dxdy.$$

Answer: $A = \sqrt{\pi}$.

Problem. 25
Compute

$$\iint_D xye^{x^2+y^2} dxdy$$

where D is the region $0 \leq x \leq y$ and $x^2 + y^2 \leq 4$.
Answer: $\frac{3e^4+1}{8}$.

Problem. 26
Compute

$$\iint_R y^2 - x^2 dxdy$$

over R defined via $0 \leq x$, $y \geq 2x$ and $x^2 + y^2 \leq 2$.

Hint: Suppose the angle between $y = 2x$ and the x-axis is α. We do not know the angle exactly but we can compute $\sin\alpha$ and $\cos\alpha$ by drawing a triangle with sides 1, 2 and $\sqrt{5}$.
Answer: $\frac{2}{5}$.

Problem. 27
(Polar/Cylindrical) Find the volume under the curve $z = 4 - x^2 - y^2$ and $z \geq 0$.
Answer: 8π.

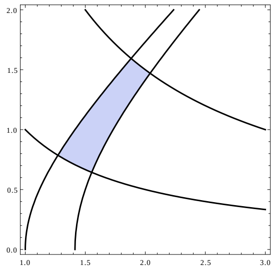
Problem. 28
Compute

$$\int_0^1 \int_0^1 \int_y^1 4z \cos(x^2) dxdydz$$

Answer: $\sin(1)$.

Problem. 29
(Polar) Compute the volume of the set (x, y, z) that satisfies $e^x + x^2 + y^2 \leq z \leq e^x + 9 - 3x^2 - 3y^2$.
Answer: $81\pi/8$.

Problem. 30
The region in the positive quadrant is enclosed by the curves $xy = 1$, $xy = 3$, $x^2 - y^2 = 1$ and $x^2 - y^2 = 2$, as shown in the picture.



Find

$$\iint_R xy(x^2+y^2)dxdy$$

Hint: Use $u = xy$, $v = x^2 - y^2$ as new coordinates and use the fact that

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}^{-1}.$$

Answer: 2.

Problem. 31

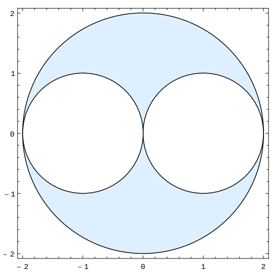
Consider the region D defined by the inequalities $\frac{1}{x} \leq y \leq \frac{2}{x}$ and $x \leq y \leq 3x$. Compute the integral

$$\iint_D \frac{2y \cdot e^{\frac{y}{x}}}{x} dxdy$$

by introducing $u = y/x$, $v = xy$.
Answer: $e^3 - e$.

Problem. 32

(Cylindrical coordinates) We have a sphere with radius 2, with two cylinders of radius 1 removed, see picture of the sphere from above. Find the volume of this set.



Answer: 128/9.

Problem. 33

Compute the generalized integral

$$\iint_{\mathbb{R}^2} \frac{4x^2+y^2}{1+(4x^2+y^2)^4} dxdy.$$

Hint: Use elliptical coordinates $x = r\cos(t)$, $y = 2r\sin(t)$. Do not forget to compute the Jacobian!
Answer: $\pi^2/8$.

Problem. 34

(Final) Compute

$$\iint_D \frac{1+x+y}{2+x-y}dxdy.$$

where D is the region bounded by the lines $x+y=-1$, $x+y=3$, $x-y=-1$ and $x-y=3$.

Answer: $\ln(625)$.

Problem. 35

Compute

$$\iint_D \frac{1}{\sqrt{y^2-x^2}}dxdy.$$

where D is the unbounded region $y^2 \leq x \leq y$.

Hint: Use the substitution $x = y\cos(t)$, and then you will need

$$\int \arccos(y)dy = y\arccos(y) - \sqrt{1-y^2} + C.$$

Answer: 1.

Problem. 36

Compute

$$\iint_D \frac{1}{xy}dxdy.$$

where D is the region $x \leq y \leq 3x$, $1 \leq xy \leq 4$ and $x, y \geq 0$.

Answer: $\ln(2)\cdot\ln(3)$.

Problem. 37

Compute

$$\iint_D xydxdy.$$

where D is the region in the first quadrant bounded by the curves $x^2+y^2=1$, $x^2+y^2=2$ $y=x^2$ and $y=x^2+1$.

Hint: Use $u = x^2+y^2$, $v = y-x^2$. The calculations are scary but doable.

Answer: $\frac{66-5\sqrt{5}-13\sqrt{13}}{48}$.

Problem. 38

Show that the integral

$$\int_0^\infty\int_0^\infty \frac{1}{1+x^3+y^3}dxdy$$

converges.

Hint: First show that for $r \geq 0$ and all t, we have that

$$r^3(\cos^3 t + \sin^3 t) \leq r^3(\cos^2 t + \sin^2 t).$$

Switch to polar and find some other function $g(r,t)$ such that it is clear that

$$\int_0^\infty\int_0^\infty \frac{1}{1+x^3+y^3}dxdy \leq \int_0^\infty\int_0^{\pi/2} g(r,\theta)rd\theta dr$$

where the latter integral can be computed easily in polar coordinates.

Problem. 39
Let D be the region $(x+y+z)^2+4(y+z)^2+4z^2 \leq 1$. Compute

$$\iiint_D x^2+y^2+z^2 dxdydz$$

Hint: Find a change of coordinates that makes the region into a sphere. Then use spherical coordinates. Some simplification can be done by considering certain symmetries.
Answer: $2\pi/15$.

Problem. 40
Compute the volume of the region $(x+y)^2+4y^2+z^2 \leq 1$.
Answer: $2\pi/3$.

Problem. 41
Compute

$$\int_0^2 \int_{3y}^{3y+6} \frac{y^2}{(1+x-3y)^3} dxdy$$

by first doing an appropriate change of variables.
Answer: 64/49.

Problem. 42
Compute the integral

$$\int_1^e \int_{1/x}^{x} \frac{\ln(x/y)\ln(xy)}{xy} dydx$$

by introducing $u = \ln(xy)$, $v = \ln(x/y)$.

Hint: Draw the xy-region and think about what uv-region this is mapped to.
Answer: 1/3.

Problem. 43
Compute the following sum

$$\int_0^2 \int_0^{\sqrt{3}x} \int_0^{\sqrt{x^2+y^2}} \arctan\left(\frac{y}{x}\right) dzdydx + \int_2^4 \int_0^{\sqrt{4^2-x^2}} \int_0^{\sqrt{x^2+y^2}} \arctan\left(\frac{y}{x}\right) dzdydx$$

by first rewriting it as a single integral in cylindrical coordinates.
Answer: The integral becomes $\int_0^4 \int_0^{\pi/3} \int_0^r \theta \cdot r \cdot dzd\theta dr = \frac{32\pi^2}{27}$.

Problem. 44
(Jacobian, Centroid) Find the centroid of the triangle with vertices $(0,0)$, $(4,3)$, $(7,0)$ in the xy-plane, $0 \leq z \leq 1$ and density $\delta(x,y,z)$ given by $1+z$.

Hint: Introduce new coordinates such that $u = 0$ on the line $(0,0)$ to $(4,3)$, and $v = 0$ on the line from $(0,0)$ to $(7,0)$.

Answer: $(\frac{11}{3}, 1, \frac{5}{9})$.

Problem. 45
(Spherical) Compute the integral

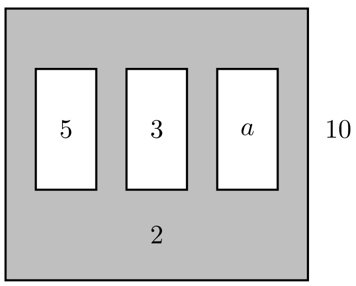
$$\iiint_D e^{-(x^2+y^2+z^2)^{3/2}} dV$$

where D is the infinite cone given by $x^2 + y^2 \leq z^2$ and $z \geq 0$.
Answer: $\frac{2\pi(1-\sqrt{2}/2)}{3}$

Problem. 46
A region E is enclosed by the curves $y = 1/x$, $y = 3/x$, $y = x$ and $y = 3x$. Find the centroid of this region, assuming uniform density.
Answer: $\left(\frac{4(5\sqrt{3}-6)}{9\ln(3)}, \frac{20-8\sqrt{3}}{3\ln(3)}\right)$

Line integrals, surface integrals, flow and flux

Problem. 47
We have a vector field $F = (M, N)$ defined in the plane, and it has continuous partial derivatives. A region D (shaded gray) with three holes is shown below. The circulation, counter-clockwise around the region D below has value 10. The value of the circulation along the three holes with orientation given by the arrows are shown written in the holes.



Finally, we are also given the fact that

$$\iint_D \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} dxdy = 2.$$

Find the value of a.
Hint: Use Greens theorem.
Answer: $a = -10$.

Problem. 48
Compute the flux across the parabola $z = 4 - x^2 - y^2$, $z \geq 0$ with normal pointing outwards, in the vector field $F = (x, y, z)$.
Answer: 24π.

Problem. 49

Compute $\iint_S F \cdot \mathbf{n} d\sigma$ where S is parametrized by

$$\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + u(1-u)\mathbf{k}, \qquad 0 \leq u, v \leq 1,$$

and F is the vector field $x\mathbf{i} + y\mathbf{j}$.
Answer: $1/6$.

Problem. 50

Find the area of the surface parametrized as

$$\mathbf{r}(u,v) = (u^2, \sqrt{2}uv, v^2), \qquad 0 \leq u, v \leq 1.$$

Answer: $\frac{4\sqrt{2}}{3}$.

Problem. 51

Compute

$$\oint_{\gamma_1} \frac{-y}{x^2+2y^2} dx + \frac{x}{x^2+2y^2} dy$$

where γ_1 is the unit circle oriented counterclockwise.

Hint:

- Show that the vector field $F = (-y, x)/(x^2 + 2y^2)$ is conservative in any region that do not contain the origin.
- Let γ_2 be the curve parametrized by $x = \cos(t)$, $y = \sin(t)/\sqrt{2}$, $0 \leq t \leq 2\pi$, and use path-independence to show that $\oint_{\gamma_1} F d\mathbf{r} = \oint_{\gamma_2} F d\mathbf{r}$.
- Compute $\oint_{\gamma_2} F d\mathbf{r}$.

Answer: $\sqrt{2}\pi$.

Problem. 52

Let $z = f(x, y)$ be a surface S over some region R in the xy-plane, with normal pointing upwards, and let $F = (M, N, P)$ be a vector field.

Show the following identity for computing the flux:

$$\iint_S F \cdot n d\sigma = \iint_R -M\frac{\partial f}{\partial x} - N\frac{\partial f}{\partial y} + P dx dy.$$

Problem. 53

Let C be the closed curve given by the intersection of the cylinder $x^2 + y^2 = 4$ and the plane $x + y + z = 5$. The curve is oriented counter-clockwise around the z-axis. Let

$$F = (e^{x^2}, \sin(y^2), 1 - x^3)$$

Find the work $\oint_C F \cdot T ds$.

Hint: Use Stokes theorem and show that the work is given by the surface integral

$$\iint_S (0, 3x^2, 0) \cdot \mathbf{n} d\sigma$$

where S is the surface parametrized as

$$\mathbf{r}(r,\theta) = (r\cos(\theta), r\sin(\theta), 5 - r\cos(\theta) - r\sin(\theta)).$$

Answer: 12π.

Problem. 54

Let C be the curve parametrized as

$$\mathbf{r}(t) = (t, t(1-t), t(1-t)), \qquad 0 \le t \le 1$$

and let $F = (x^2 + e^{x^2}(z-y), \sin(y^2), \cos(z^2))$. Compute the work along C.

Hint: Add the line L from $(1,0,0)$ to $(0,0,0)$ such that $C \cup L$ is a simple closed loop that encloses a region in the plane $y = z$. Use Stokes theorem to show that

$$\int_C F d\mathbf{r} = -\int_L F d\mathbf{r}$$

and then finally compute $-\int_L F d\mathbf{r}$. Be careful with normals and orientations of curves.
Answer: $\frac{1}{3}$.

Problem. 55

Let S be surface of the paraboloid $z = 4 - x^2 - y^2$, $z \ge 0$ with outward-pointing normal, and let $F = (-x^2y, xy^2, ze^{x^2+y^2})$. Compute the surface integral

$$\iint_S (\nabla \times F) \cdot \mathbf{n} d\sigma$$

by converting it to a line integral via Stokes theorem.

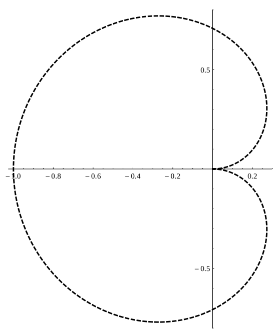
Hint: You need to close the surface by adding a disk before applying Stokes.
Answer: 8π.

Problem. 56

Consider curve parametrized as

$$\mathbf{r}(t) = (\sin(t)\cos(2t), \sin(t)\sin(2t)), \qquad 0 \le t \le \pi.$$

This is a simple closed curve with a counter-clockwise orientation. Compute the area of the region.



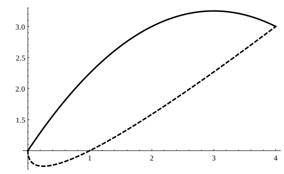
Answer: $\frac{\pi}{2}$.

Problem. 57

We are given two curves:

$$\mathbf{r}_1 = (t^2, (t-1)^2 + t), \qquad \mathbf{r}_2 = (2t, 1 - t^2 + 3t), \qquad 0 \le t \le 2.$$

They enclose a simple closed region (see figure). Compute the area of the region.



Hint: Be careful with orientation of the curves.
Answer: 4.

Problem. 58

Let $x^2 + y^2 = z$ define a parabola with normal pointing outwards. Let S be the part of the parabola which lies between the planes $z = 9$ and $z = 2x$. Consider the field $F = ((2x - z)e^{z^2}, y, -y)$ and compute the integral

$$\iint_S (\nabla \times F)\mathbf{n}d\sigma.$$

Hint: Use Stokes theorem. You can see the surface integral as a difference of two integrals: let S_1 be the points on the parabola below the plane $z = 9$, and let S_2 be everything on the parabola below $z = 2x$. Then $S = S_1 - S_2$. See also Exercise 47.
Answer: 2π.

Mixed problems

Problem. 59

Let $z = x \cdot h(x, y) + xy^2$, where $x = 2t^2 + \cos(\pi t)$, $y = \sin(2\pi t)$ and we know that

$$h(1,0) = 3, \quad h'_x(1,0) = 2, \quad h'_y(1,0) = \frac{1}{\pi}.$$

Find $\frac{dz}{dt}$ at $t = 1$.
Answer: 22.

Problem. 60

Determine k such that the vector $(2k, 3 + k, -1)$ is parallel to the plane $2x - y + 3z = 4$.

Answer: $k = 4/3$.

Problem. 61

Determine k such that the volume spanned by the vectors $(6, k, 0)$, $(3, 4, -k)$, $(1, 2, 2)$ is maximized.

Answer: $k = 3$.

Solution: The determinant of these vectors is $48 + 6k - k^2$.

Problem. 62

There are two points on the curve $\mathbf{r}(t) = (t+3)\mathbf{i} + (t^2 - 4t + 4)\mathbf{j} + (t-4)\mathbf{k}$ that also lies in the plane $2x + y - z = 14$. Let P be the one that is nearest the origin.

In P, determine the (short) angle between the normal of the plane, and the tangent vector of the curve at P.

Answer: $\pi/3$.

Solution: The values of t that yields points on the plane are solutions to

$$\begin{aligned} 2(t+3) + (t^2 - 4t + 4) - (t-4) = 14 \\ t^2 - 3t + 14 = 14. \end{aligned}$$

The point is common with the plane occurs when $t = 0$ and $t = 3$. We have

$$\mathbf{r}(0) = (3, 4, -4), \qquad \mathbf{r}(3) = (6, 1, -1).$$

The point $(6, 1, -1)$ is nearest the origin. Now, $\mathbf{r}'(3) = (1, 2, 1)$. The normal of the plane is $(2, 1, -1)$ and

$$\frac{(1,2,1) \cdot (2,1,-1)}{\sqrt{1+4+1}\sqrt{4+1+1}} = \frac{2+2-1}{\sqrt{6}\sqrt{6}} = \frac{1}{2}$$

The angle is therefore $\pi/3$.

Problem. 63

Find a vector v that is perpendicular to both $(1, 0, 1)$ and $(1, 2, -1)$, and such that the vectors v and $(0, 1, -1)$ span a parallelogram with area $5\sqrt{6}$.

Answer: $v = \pm(-5, 5, 5)$.

Solution: Taking cross product of the first two vectors implies that v is parallel with $(-1, 1, 1)$. Since $(-1, 1, 1)$ and $(0, 1, -1)$ are also orthogonal, the area spanned by $a(-1, 1, 1)$ and $(0, 1, -1)$ is $a\sqrt{3} \cdot \sqrt{2}$. Thus, $a = \pm 5$.

Problem. 64

Let $\mathbf{r}(t) = (t^2 \sin(2t), t^2 \cos(2t), \sqrt{3}t^2)$. Find the arc-length of $\mathbf{r}(t)$ from -2 to 2.

Answer: $32(2\sqrt{2} - 1)/3$.

Solution: The integral becomes $2\int_0^2 2t\sqrt{4+t^2}dt$ which can be solved by a simple u-substitution.

Problem. 65

A ball with horizontal velocity $5m/s$ rolls off a window sill $20m$ above the ground. The ball bounces on the ground and retains 25% of its speed in the vertical direction.

How far away from the building wall when it hits the ground the second time? Assume a gravity of $10m/s^2$.

Answer: $15m$

Solution: Before the bounce, the height above ground is $h(t) = 20 - 5t^2$, so it hits the ground after 2 seconds. At that time, the vertical speed is $20m/s$, and it then bounces with speed $5m/s$.

After the bounce, the height function is $h_2(t) = 5t - 5t^2 = 5t(1-t)$, so it hits the ground a second time 1 additional second from the start. Thus, it has traveled $5(2+1) = 15m$.

Problem. 66

Let $\mathbf{r}(t) = (e^t \cos(t), e^t \sin(t), e^t)$. Find the principal unit normal vector of the curve at $t = \pi$.

Answer: $(-1, 1, 0)/\sqrt{2}$.

Problem. 67

Find the normal component of the acceleration of the curve $r(t) = (1, t, t^2)$ at $t = -1$.

Answer: $a_N(-1) = 2/\sqrt{5}$.

Problem. 68

Find the curvature of $\mathbf{r}(t) = (\sqrt{4-t}, \sqrt{t})$ for $t = \pi$.

Answer: $1/2$.

Solution: The curvature is $1/2$ everywhere — the curve is a quarter-circle with radius 2. It is straightforward to find $T(t) = \frac{1}{2}(-\sqrt{t}, \sqrt{4-t})$, and this is very similar to two times $\mathbf{r}(t)$.

It follows that $|T'(t)|/|\mathbf{r}'(t)| = \frac{1}{2}$.

Midterm problems

Problem. 69

A plane given by $x + 3y - 2z = 2$ intersects a line parametrized by $(2, 0, 0) + (-1, 0, 1)t$. Find the x-coordinate of the intersection point.

Solution. 69

Plug in the line parametrization into the equation of the plane: $(2-t) + 3(0) - 2(t) = 2$. Solving this for t gives $t = 0$. We plug $t = 0$ into the line parametrization and get the point $(2, 0, 0)$. The x-coordinate is 2.

Problem. 70

Find a value of k such that if $\mathbf{u} = (3, 0, 1)$, $\mathbf{v} = (0, 2, 1)$ and $\mathbf{w} = (k, 2, 2)$, then $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = 0$.

Solution. 70

We compute the triple scalar product using the determinant formula:

$$\begin{vmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \\ k & 2 & 2 \end{vmatrix} = 3\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} + 1\begin{vmatrix} 0 & 2 \\ k & 2 \end{vmatrix} + 3(4-2) - 2k = 6 - 2k$$

This is 0 if $k = 3$.

Problem. 71

A plane has the equation $2x + y - z = 2$. Find the distance from $(5, 0, 0)$ to the plane.

Solution. 71

We begin with finding a point in the plane: $P = (1, 0, 0)$. Furthermore, the normal of the plane is $\mathbf{n} = (2, 1, -1)$. Let $S = (5, 0, 0)$. The distance formula gives the distance

$$\frac{|PS \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{(4,0,0)\cdot(2,1,-1)}{\sqrt{4+1+1}} = 8/\sqrt{6}.$$

Problem. 72

The equations

$$1)\ x^2 + 4y^2 = 6 \qquad 2)\ 4x^2 + 4z^2 = y \qquad 3)\ 3x^2 + 5z^2 + 3y^2 = 6$$

defines conic surfaces in space. One is an ellipsoid (E), one is a paraboloid (P) and one is a cylinder (C). Find the correct matching.

$a)$ $1C$-$2E$-$3P$ $\quad$ $b)$ $1E$-$2C$-$3P$ $\quad$ $c)$ $1P$-$2C$-$3E$

$d)$ $1C$-$2P$-$3E$ $\quad$ $e)$ $1E$-$2P$-$3C$ $\quad$ $f)$ $1P$-$2E$-$3C$

Solution. 72

We have, in order, a cylinder, a paraboloid and an ellipsoid: $1C - 2P - 3E$.

Problem. 73

A particles position is given by the curve $\mathbf{r}(t) = (2t, t^2 - 1, 2 + 2t)$. At a certain point P on the curve, the tangent vector is parallel with the plane $2x + 3y - z = 2$. What is the x-coordinate of P?

Solution. 73

To be parallel with a plane is the same as being perpendicular to the normal of the plane. The tangent line direction is given by $\mathbf{r}'(t) = (2, 2t, 2)$. Thus, we seek t such that $\mathbf{r}'(t) \cdot (2, 3, -1) = 0$. We get the equation $4 + 6t - 2 = 0$, which is solved by $t = -1/3$.

The x-coordinate of $\mathbf{r}(-1/3)$ is $-2/3$.

Problem. 74

A golf ball is hit in an upwards angle of $60°$ and lands 5 seconds later. How fast across the ground did the ball travel in meter per second?

Use $g = 10$ meter per second^2.

Solution. 74

The initial velocity of the ball is given by $s(\cos 60°, \sin 60°) = s(1/2, \sqrt{3}/2)$, where s is the initial speed. Integrating and adding the gravity contribution gives the position

$$\mathbf{r}(t) = s(1/2, \sqrt{3}/2)t - (0, 5)t^2.$$

After 5 seconds, we know that the y-coordinate is 0. Plugging in $t = 5$ in the y-component gives

$$5\sqrt{3}s/2 - 5 \cdot 5^2 = 0$$

so $s = 50/\sqrt{3}$. This is the initial *speed*, but we seek the x-component. This is given by $(50/\sqrt{3})\cos 60° = 25/\sqrt{3}$.

Problem. 75
A curve is parametrized as $\mathbf{v}(t) = e^t(\cos 2t, \sin 2t, 1)$, for $0 \leq t \leq 1$. Find the length of the curve.

Solution. 75
We compute

$$\mathbf{v}'(t) = (e^t\cos(2t) - 2e^t\sin(2t), e^t\sin(2t) + 2e^t\cos(2t), e^t).$$

Then, $|\mathbf{v}'(t)|$ is given by

$$e^t\sqrt{(\cos(2t) - 2\sin(2t))^2 + (\sin(2t) + 2\cos(2t))^2 + 1^2} = e^t\sqrt{(5(\cos^2(2t) + \sin^2(2t)) + 1} = e^t\sqrt{6}$$

The arc-length is then given by

$$\int_0^1 |\mathbf{v}'(t)|dt = \int_0^1 e^t\sqrt{6}dt = \sqrt{6}(e^1 - e^0) = \sqrt{6}(e - 1).$$