Math 114
Fall 2016
Midterm 4 - Take home
Due Dec 9

Name (Print): ____________________

Recitation section: ____________________

This exam contains 8 pages (including this cover page) and 6 problems. Enter all requested information on the top of this page, and put your name on the top of every page, in case the pages become separated.

You are required to show your work on each problem unless stated otherwise.

- Since this is a take home exam, I expect you to sketch a solution on scratch paper first, and only write down your *final* solutions *neatly* on this exam.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	12	
3	12	
4	12	
5	12	
6	12	
Total:	70	

1. (a) (5 points) Let $z = f(x, y)$ be a smooth surface, and let C be part of a level curve of f. Furthermore, assume that $|\nabla f| = 3$ everywhere on C, and that C has length 5. Compute the following line integrals:

$$\int_C (\nabla f) \cdot \mathbf{n} ds = \underline{\hspace{3cm}}, \qquad \text{and} \qquad \int_C (\nabla f) \cdot d\mathbf{r} = \underline{\hspace{3cm}}$$

Hint: Draw a picture!

Solution: We have that ∇f and $\mathbf{n}$ are parallel, so $(\nabla f) \cdot \mathbf{n} = |\nabla f||\mathbf{n}| = 3$ everywhere on the curve. The first integral is then $3 \int_C ds = 3 \cdot 5 = 15$.

In the second integral, we integrate $(\nabla f) \cdot \mathbf{T}$. This quantity is 0, since the vectors ∇f and $\mathbf{T}$ are perpendicular everywhere om C. The second integral is therefore 0.

Alternatively, in the second integral, since ∇f is conservative, f is the potential function, and it is enough to compute $f(b) - f(a)$, (the endpoints of the curve). Since C is a level curve, $f(b) = f(a)$.

(b) (5 points) Let $F = (M, N, P)$ be a vector field, with continuous partial derivatives. Let S_θ be the plane through the origin with unit normal vector $\mathbf{n} = (\cos\theta, \sin\theta, 0)$, and let C_θ be the disk given by the intersection of S_θ and the unit ball, oriented counter-clockwise around the unit normal vector $\mathbf{n}$.

In other words, C_0 and C_π are the same disks, but we consider them to have normals in opposite directions.

If the flux $\iint_{C_0} F \cdot \mathbf{n} d\sigma = 5$, what is $\iint_{C_\pi} F \cdot \mathbf{n} d\sigma$?

Argue that there is some θ such that the total flux across C_θ is 0.

Solution: Let $f(\theta) = \iint_{C_\theta} F \cdot \mathbf{n} d\sigma$. We know that $f(0) = 5$, and rotating the disk half a turn changes the orientation of the boundary. Hence, $f(\pi) = -5$. Since the region is smooth, and F has nice properties, $f(\theta)$ is a continuous function. The intermediate value theorem now gives that there is some θ^* such that $f(\theta^*) = 0$.

Comment: Note that for each θ, the flux is given as an integral over a disk. Thus, we *cannot* say that there is a θ where "F is perpendicular to $\mathbf{n}$" — this sentence does not make sense. The angle between F and $\mathbf{n}$ depend on where on the disk you are. In some parts of the disk $C\theta$, $F \cdot \mathbf{n}$ might be positive, on other parts, negative. We seek a θ such that the *sum over all these scalar products* on C_θ is 0. We cannot guarantee anything better.

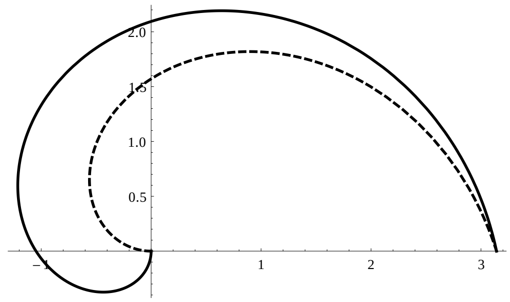
2. (12 points) *Greens theorem for area* states that a region R whose boundary is a simple closed curve C, satisfies

$$area(R) = \frac{1}{2}\oint_C x dy - y dx.$$ See p. 991 in the book.

Find the area of the region enclosed by the curves C_1 (dashed) and C_2 (solid) , whose parameterizations are given by

$$C_1 : (-t\cos(t), t\sin(t)), \qquad C_2 : (-t\sin(3t/2), -t\cos(3t/2)), \qquad 0 \le t \le \pi.$$

$a)\ \pi^3/12 \qquad b)\ \pi^2/8 \qquad c)\ \pi^2/6 \qquad d)\ \pi^2/4 \qquad e)\ \pi/6 \qquad f)\ \pi^2/2$



Solution: Note first that we need to traverse the curve counter-clockwise in order to Green's theorem to apply. The curve C_2 is given with the wrong orientation (it starts at the origin), so we need to remember to subtract this.

First task is to construct the expression $xdy - ydx$ for each curve.

Curve 1:

$$\mathbf{r}_1(t) = (-t\cos(t), t\sin(t)), \qquad \mathbf{r}_1'(t) = (t\sin(t) - \cos(t), \sin(t) + t\cos(t))$$

This gives that

$$xdy - ydx = (-t\cos(t))[\sin(t) + t\cos(t)] - (t\sin(t))[t\sin(t) - \cos(t)] = -t^2.$$

Curve 2:

$$\mathbf{r}_2(t) = (-t\sin(3t/2), -t\cos(3t/2)),$$
$$\mathbf{r}_2'(t) = \left(-\sin\left(\frac{3t}{2}\right) - \frac{3}{2}t\cos\left(\frac{3t}{2}\right), \frac{3}{2}t\sin\left(\frac{3t}{2}\right) - \cos\left(\frac{3t}{2}\right)\right)$$

This gives (after simplification)

$$xdy - ydx = -\frac{3t^2}{2}.$$

Now we remember that we need to take the reverse orientation of C_2, meaning we need to change the sign. The area is therefore given by

$$\frac{1}{2}\int_0^{\pi} -t^2 dt + \frac{1}{2}\int_0^{\pi} \frac{3t^2}{2} dt = \frac{1}{2}\int_0^{\pi} \frac{t^2}{2} dt = \left[\frac{t^3}{12}\right]_0^{\pi} = \frac{\pi^3}{12}.$$

3. (12 points) Compute the line integral

$$\int_C \left(\frac{x-y}{x^2+y^2}+1\right)dx + \left(\frac{x+y}{x^2+y^2}+1\right)dy$$

along the following two curves:

(I) The line from $(1,0)$ to $(0,1)$.

(II) The curve consisting of the two lines $(1,0)$ to $(-1,-1)$ and $(-1,-1)$ to $(0,1)$.

Hint: Show first that the vector field is conservative in its domain.

a) $\frac{\pi}{2}$ and $-\frac{3\pi}{2}$ *b*) $\frac{\pi}{2}$ and $\frac{\pi}{2}$ *c*) $\frac{\pi}{2}$ and 0

d) 0 and $\frac{\pi}{2}$ *e*) 0 and 0 *f*) $\frac{\pi}{4}$ and 2π

Solution: We check if the vector field is conservative — $\partial M/\partial y = \partial N/\partial x$. After some calculation, we see that this is indeed the case. The domain of F is all points in $\mathbb{R}^2$ except $(0,0)$ where it is not defined.

This means that the integrals are path-independent as long as when we deform the path, the region between the old and the new path does not contain $(0,0)$.

Let $\gamma_1(t) = (\cos(t), \sin(t))$, $0 \le t \le \pi/2$. This quarter-circle has $(1,0)$ as starting point and $(0,1)$ as endpoint, and the region between this curve and (I) does not contain $(0,0)$.

The first line integral therefore has the same value as

$$\int_0^{\pi/2} \left(\frac{\cos(t)-\sin(t)}{\cos^2(t)+\sin^2(t)}+1\right)(-\sin(t))dt + \left(\frac{\cos(t)+\sin(t)}{\cos^2(t)+\sin^2(t)}+1\right)(\cos(t))dt$$

This simplifies to

$$\int_0^{\pi/2} \sin^2(t) - \sin(t) + \cos^2(t) + \cos(t)dt = \int_0^{\pi/2} 1 + \cos(t) - \sin(t)dt = \frac{\pi}{2}.$$

For the curve (II), we use a similar reasoning, and deform the curve to

$$\gamma_2(t) = (\cos(t), -\sin(t)), \qquad 0 \le t \le 3\pi/2.$$

which is a three-quarter circle starting at $(1,0)$, ending at $(0,1)$, and traversing clockwise around the origin. The second integral is then

$$\int_0^{3\pi/2} \left(\frac{\cos(t)+\sin(t)}{\cos^2(t)+\sin^2(t)}+1\right)(-\sin(t))dt + \left(\frac{\cos(t)-\sin(t)}{\cos^2(t)+\sin^2(t)}+1\right)(-\cos(t))dt$$

which simplifies to

$$\int_0^{3\pi/2} -\sin^2(t) - \sin(t) - \cos^2(t) - \cos(t)dt = \int_0^{3\pi/2} -1 - \cos(t) - \sin(t)dt = -\frac{3\pi}{2}.$$

4. (12 points) Compute the line integral of $F = (x, e^{y^2}, 3xy)$ along the curve C, whose segments are given as

$$C_1 = (0, t, t^2), \qquad C_2 = (t, 1, 1), \qquad C_3 = (1, 1-t, (1-t)^2)$$

all when $0 \leq t \leq 1$, starting from $(0,0,0)$ and ending at $(1,0,0)$.

Hint: By adding the line from $(1,0,0)$ to $(0,0,0)$ and choosing an appropriate surface, you can use Stokes theorem.

a) $-1/2$ *b*) -1 *c*) $-3/2$ *d*) -2 *e*) $-5/2$ *f*) -3

Solution: The only difference between C_1 and C_3 is that they have different x-coordinates. Thus,

$$\mathbf{r}(u,v) = (u, v, v^2), \qquad 0 \leq u \leq 1, \quad 0 \leq v \leq 1$$

parametrizes a surface whose boundaries are C_1, C_2, C_3 and C_4, the straight line from $(1,0,0)$ to $(0,0,0)$.

In order for Stokes to apply, we need the normal to the surface to be pointing down, as C_1, C_2, C_3 and C_4 then would be traversed in a counter-clockwise fashion. We now have

$$\mathbf{r}_u \times \mathbf{r}_v = (1,0,0) \times (0,1,2v) = (0,-2v,1)$$

which is pointing up. We flip the sign, compute the curl, $(\nabla \times F) = (3x, -3y, 0)$, and get that the integral along all four curves is (via Stokes):

$$\int_0^1 \int_0^1 (3u, -3v, 0) \cdot (0, 2v, 1) du dv = \int_0^1 \int_0^1 -6v^2 du dv = -2$$

We now have to subtract the line integral of the line from $(1,0,0)$ to $(0,0,0)$. We reverse the direction and add it instead. The line is parametrized as $(t,0,0)$, $0 \leq t \leq 1$. We get

$$\int_0^1 (t, e^0, 0) \cdot (1,0,0) dt = \int_0^1 t dt = \frac{1}{2}.$$

The answer is therefore $-2 + \frac{1}{2} = -\frac{3}{2}$.

Alternatively: Another way to compute the surface integral, is to realize that the surface lies above the square $0 \leq x, y \leq 1$ in the xy-plane, and is described implicitly $g(x,y,z) = y^2 - z = 0$. This surface has normal $\pm\nabla g/|\nabla g|$. We wish to integrate $G(x,y,z) = \nabla \times F = (3x, -3y, 0)$ over this surface. One can then use the formula for such surface integrals (here $\mathbf{p} = (0,0,1)$, the normal of the xy-plane):

$$\iint_S G(x,y,z) \frac{\nabla g}{|\nabla g \cdot \mathbf{p}|} dx dy = \int_0^1 \int_0^1 (3x, -3y, 0) \cdot \frac{(0, 2y, -1)}{|1|} dx dy = \int_0^1 \int_0^1 -6y^2 dx dy$$

where we picked a sign such that the normal is pointing down (such that the path is counter-clockwise around the normals). This integral is, of course, -2.

5. (12 points) Consider the box B in $0 \leq x, y, z \leq 1$, with *the top square removed*, and all other surface normals are pointing outwards. Let $F = (e^{y^2}, y^2 z, x - y + z)$.

   Compute the total flux given by F through the five sides of B.

   a) 1/2 *b)* 1 *c)* 3/2 *d)* 2 *e)* 5/2 *f)* 3

Solution: We close the cube by adding the top. Then we may use the divergence theorem on the cube D.

$$flux(D) = \iiint_D \nabla \cdot F dV = \int_0^1 \int_0^1 \int_0^1 0 + 2yz + 1 dxdydz = \frac{3}{2}$$

We now need to subtract the flux though the top T. The top is parametrized as $(x, y, 1)$, $0 \leq x, y \leq 1$ and the normal is $(0, 0, 1)$ since it must be pointing out from the cube. We have

$$flux(T) = \iint_T F \cdot \mathbf{n} d\sigma = \int_0^1 \int_0^1 (e^{y^2}, y^2, x - y + 1) \cdot (0, 0, 1) dxdy = \int_0^1 \int_0^1 x - y + 1 dxdy = 1$$

Hence, the flux though the five sides is $\frac{3}{2} - 1 = \frac{1}{2}$.

6. (12 points) Let $\delta(x,y,z) = \frac{9x^2+24xy+16y^2}{25z}$ be the density function for a thin surface S, that is parametrized as

$$\mathbf{r}(u,v) = \frac{3u+4v}{5}\mathbf{i} + \frac{4u-3v}{5}\mathbf{j} + u^2\mathbf{k}, \qquad 0 \le u \le 1, \quad 0 \le v \le u.$$

Find the mass of the surface, and the z-coordinate of its center of mass.

a) $M = \frac{5\sqrt{5}-1}{12}$, $\bar{z} = \frac{313+15\sqrt{5}}{620}$ *b)* $M = \frac{2\sqrt{5}+\ln(2+\sqrt{5})}{4}$, $\bar{z} = \frac{313-15\sqrt{5}}{120}$

c) $M = \frac{5\sqrt{5}-1}{6}$, $\bar{z} = \frac{781+30\sqrt{5}}{2171}$ *d)* $M = \frac{2\sqrt{5}+\ln(2+\sqrt{5})}{4}$, $\bar{z} = \frac{1}{2}$

Solution: If we plug in the parametrization in the density, we see that the surface has constant density 1. Furthermore,

$$\mathbf{r}_u = (3/5, 4/5, 2u), \qquad \mathbf{r}_v = (4/5, -3/5, 0) \qquad \mathbf{r}_u \times \mathbf{r}_v = (6u/5, 8u/5, -1)$$

It follows that $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{1+4u^2}$.

The mass M is then

$$\int_0^1 \int_0^u \sqrt{1+4u^2}\,dv\,du = \int_0^1 u\sqrt{1+4u^2}\,du = \frac{5\sqrt{5}-1}{12}.$$

where the integral can be treated using $t = 1+4u^2$ as a substitution.

The moment about the xy-plane is given by

$$\int_0^1 \int_0^u z\sqrt{1+4u^2}\,dv\,du = \int_0^1 \int_0^u u^2\sqrt{1+4u^2}\,dv\,du = \int_0^1 u^3\sqrt{1+4u^2}\,du$$

We do the same substitution as before, and get after some calculation, $M_{xy} = (25\sqrt{5}+1)/120$.

Finally,

$$\bar{z} = \frac{M_{xy}}{M} = \frac{5\sqrt{5}-1}{12} \cdot \frac{120}{25\sqrt{5}+1}$$

which simplifies to $\frac{313+15\sqrt{5}}{620}$.