Math 114
Fall 2016
Midterm 3a
16/11/16
Time Limit: 50 Minutes

Name (Print): ____________________

Recitation section: ____________________

This exam contains 6 pages (including this cover page) and 4 problems. Enter all requested information on the top of this page, and put your name on the top of every page, in case the pages become separated.

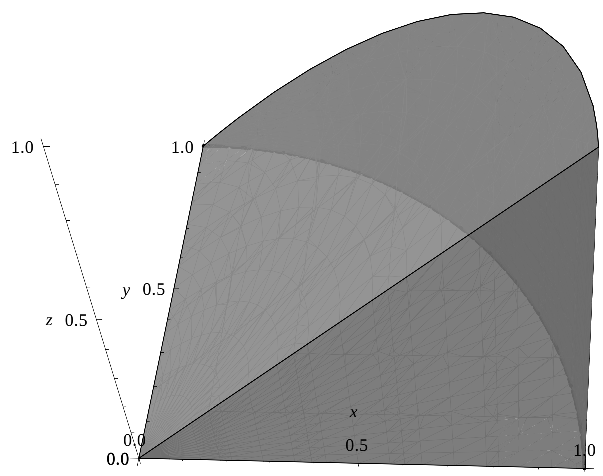
You are required to show your work on each problem unless stated otherwise.

- There are some **useful formulas on the extra page**!
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- You must bring your PennID and have it out during the exam as someone could around to do an ID check.
- Once you finish the exam you must remain seated until the time has expired and your exam has been collected.

Problem	Points	Score
1	12	
2	18	
3	18	
4	22	
Total:	70	

Do not write in the table to the right.

1. (12 points) The base of a solid E is the quarter unit-disk in the first quadrant of the xy-plane, and it is bounded by the planes $z = 0$ and $z = x$. Which integrals compute the volume of E?



Check all correct:

☐ $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^x dzdydx,$

☐ $\int_0^1 \int_0^{\sqrt{1-y^2}} x \cdot dxdy$

☐ $\int_0^1 \int_0^{\pi/2} \int_0^{r\cos\theta} r \cdot dzd\theta dr$

☐ $\int_0^1 \left(\int_0^x dy \right) \left(\int_0^{\sqrt{1-x^2}} dz \right) dx$

☐ $\int_0^{\pi/4} \int_0^{1/\cos\theta} \int_0^{\sqrt{1-r^2\cos^2\theta}} r \cdot dydrd\theta$

☐ $\int_0^{\pi/2} \int_0^1 \int_0^{r\cos\theta} r \cdot dzdrd\theta$

You do not need to show your work on this problem.

Solution: All alternatives parametrizes the region.

2. (18 points) Compute

$$\iint_D x^2 - y^2 dxdy$$

over the region D in the first quadrant given by $1 \leq xy \leq 5$, $1 \leq x - y \leq 2$, by choosing an appropriate change of coordinates.

$a)$ 1 $\quad b)$ 2 $\quad c)$ 3 $\quad d)$ 4 $\quad e)$ 6 $\quad f)$ -6

Solution: We use the substitution $u = xy$, $v = x - y$. Then,

$$J(x,y) = \begin{pmatrix} y & x \\ 1 & -1 \end{pmatrix} = -(x+y)$$

In the region, both variables are positive, so $|J(x,y)| = x + y$. It is easy to see what bounds we get, and the integral becomes

$$\int_1^5 \int_1^2 \frac{x^2 - y^2}{x+y} dvdu = \int_1^5 \int_1^2 \frac{(x+y)(x-y)}{x+y} dvdu =$$

$$\int_1^5 \int_1^2 vdvdu = \int_1^5 du \cdot \int_1^2 vdv = 4 \cdot \frac{(2^2 - 1^2)}{2} = 6.$$

Technically, we should argue that this map $u = xy$, $v = x - y$ is invertible — this is not clear since the map is not linear. From the second equation, we have $y = x - v$. Inserted in the first equation gives $u = x(x - v)$, so

$$x^2 - vx - u = 0 \qquad \Longrightarrow \qquad x = \frac{v}{2} \pm \sqrt{v^2/4 + u}$$

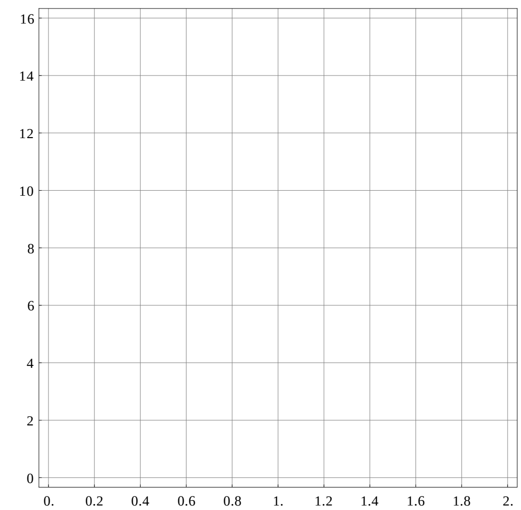
Since we know $x > 0$, and $\sqrt{v^2/4 + u} > v/2$ as long as $u > 0$, we pick the positive solution. Once we know x, it is easy to find y in terms of u and v, which shows that the map is invertible.

3. (18 points) Compute

$$\int_1^4 \int_1^{\sqrt{y}} e^{x^3}\, dx dy + \int_4^{16} \int_{\sqrt{y}/2}^{2} e^{x^3}\, dx dy.$$

Hint: Draw the regions of the integrals and change order of integration.

$a)\ e^2 - e \qquad b)\ e^2 - 1 \qquad c)\ e^8 - e \qquad d)\ e^7 \qquad e)\ e^2 + 1 \qquad f)\ 2978$



Solution: Drawing the regions, it is clear that we can write the sum as a single integral:

$$\int_1^2 \int_{x^2}^{4x^2} e^{x^3}\, dy dx = \int_1^2 (4x^2 - x^2) e^{x^3}\, dx = \int_1^2 3x^2 e^{x^3}\, dx = \left[e^{x^3} \right]_{x=1}^{2} = e^8 - e.$$

4. (22 points) Consider a thin plate in the unbounded region $0 \leq y \leq x$, with density given by $\delta(x, y) = \arctan(\frac{y}{x})e^{-\sqrt{x^2+y^2}}$. Find the y-coordinate of the center of mass.

   Partial credit for the following steps:

   a) Express the mass of the region as an integral in polar coordinates.

   b) Compute the mass M of the region.

   c) Compute the first moment M_x.

a) $\sqrt{2}$ *b)* 1 *c)* $\frac{8\sqrt{2}-\pi}{\pi^2}$ *d)* $\frac{\sqrt{2}+\pi}{\pi^2}$ *e)* $\frac{\sqrt{2}+1}{\pi^2}$ *f)* $\frac{8\sqrt{2}(4-\pi)}{\pi^2}$

Solution: The unbounded region in polar coordinates is $0 \leq \theta \leq \pi/4$ and $0 \leq r$. The only slightly tricky part of the integrand is the $\arctan(y/x)$. In polar coordinates, we get

$$\arctan\left(\frac{r\sin\theta}{r\cos\theta}\right) = \arctan(\tan\theta) = \theta.$$

So, in polar coordinates, we need to compute the integrals

$$M = \int_0^\infty \int_0^{\pi/4} \theta e^{-r} r d\theta dr, \qquad M_x = \int_0^\infty \int_0^{\pi/4} (r\sin\theta)\theta e^{-r} r d\theta dr,$$

We start with the first integral, that separates as

$$M = \int_0^{\pi/4} \theta d\theta \cdot \int_0^\infty re^{-r} dr = \frac{\pi^2}{32} \cdot \left[-re^{-r} - e^{-r}\right]_0^\infty = \frac{\pi^2}{32}.$$

The second integral is a bit more of a challenge:

$$M_y = \int_0^{\pi/4} \theta\sin(\theta) d\theta \cdot \int_0^\infty r^2 e^{-r} dr$$

The trick is to do integration by parts in both of these cases. We get

$$[\sin(\theta) - \theta\cos(\theta)]_{\theta=0}^{\pi/4} \cdot \left[-(2 + 2r + r^2)e^{-r}\right]_0^\infty = \left(\frac{4-\pi}{4\sqrt{2}}\right) \cdot 2$$

Then,

$$\bar{x} = \frac{(4-\pi)/(2\sqrt{2})}{\pi^2/32} = \frac{8\sqrt{2}(4-\pi)}{\pi^2}.$$

Extra page

$$\int \theta \sin\theta d\theta = \sin\theta - \theta\cos\theta + C, \qquad \int \theta \cos\theta d\theta = \cos\theta + \theta\sin\theta + C,$$

$$\int re^r dr = (r-1)e^r + C, \qquad \int re^{-r} dr = -(r+1)e^{-r} + C,$$

$$\int r^2 e^r dr = (r^2 - 2r + 2)e^r + C, \qquad \int r^2 e^{-r} dr = -(r^2 + 2r + 2)e^{-r} + C.$$