Math 114
Fall 2016
Midterm 2a
26/10/16
Time Limit: 50 Minutes

Name (Print): ____________________

Recitation section: ____________________

This exam contains 10 pages (including this cover page) and 7 problems. Enter all requested information on the top of this page, and put your name on the top of every page, in case the pages become separated.

You are required to show your work on each problem unless stated otherwise.

- If you need more space, use the back of the pages; clearly indicate when you have done this.
- You must bring your PennID and have it out during the exam as someone could around to do an ID check.
- Once you finish the exam you must remain seated until the time has expired and your exam has been collected.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	8	
5	12	
6	10	
7	10	
Total:	70	

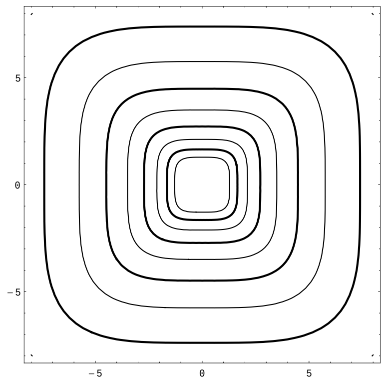
1. For these questions, you do not need to show your work — only provide an answer.

   (a) (3 points) The directional derivative of a function in direction $(\sqrt{2}, \sqrt{2})/2$ in a point P_0 might exist even if the function does not have a gradient in P_0.

   True / **False**.

   Solution: True. E.g. $f(x, y) = 1$ if $x = y$ and 0 otherwise.

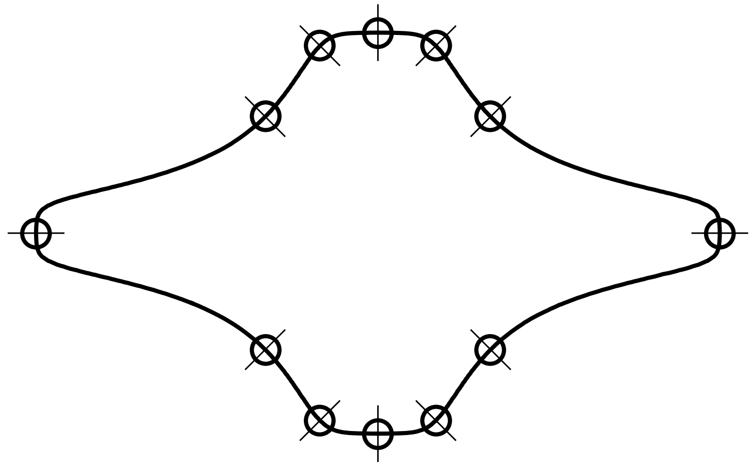
   (b) (3 points) The level curves below correspond to which function?



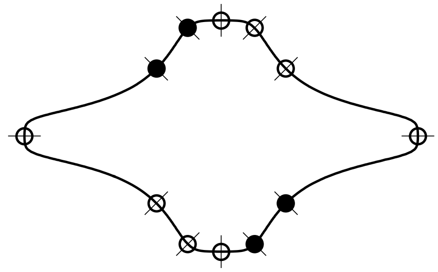
   Underline one: $\ln(xy)$, $|x^3 + y^3|$, $\ln(x^4 + y^4)$, $x + y$, $\arctan(xy)$, $\frac{xy}{x^2+y^2}$.

   Solution: It is $\ln(x^4 + y^4)$. The functions $\arctan(xy)$, $\frac{xy}{x^2+y^2}$ are constant on the x-axis, but this is not a level curve in the figure. Functions $|x^3 + y^3|$ and $x + y$, are constant on $x = -y$, which we do not see. Finally, $\ln(xy)$ is not even defined in two of the quadrants, so we rule this out as well.

   (c) (4 points) You want to find the maximum of $f(x, y) = -x + y + 100$, under the condition $g(x, y) = 0$ for a complicated function g. The set $g(x, y) = 0$ is drawn in the xy-plane below. The method of Lagrange multipliers produces a set of *critical points*, where ∇f is parallel with ∇g. **Clearly mark these critical points in the picture.**



Solution: The gradient ∇g is perpendicular to the curve in the figure, and $\nabla f = (-1, 1)$. Thus, we mark the points with the gradient pointing in the direction of $(-1, 1)$ or $(1, -1)$.



2. For these questions, correct answer gives full score. Some points might be given for partial work.

(a) (2 points) Let $f(x, y)$ be differentiable, and suppose that in $(0, 0)$, the gradient vanish, and $f''_{xx} = 5$, $f''_{yy} = 2$ and $f''_{xy} = -3$. Characterize the critical point in $(0, 0)$.
Underline one: Local max, local min, saddle-point, inconclusive.

Solution: The formula tells us that the Hessian is $5 \cdot 2 - (-3)^2 = 1$. This together with $f''_{xx} > 0$ tells us it is a local minimum.

(b) (3 points) Suppose $\nabla f = (3z - yz, -xz, 3x - xy)$ and $f(1, 1, 1) = 2$. Use the standard linear approximation of f in $(1, 1, 1)$ to estimate $f(1.2, 1.1, 0.9)$.
Answer: $f(1.2, 1.1, 0.9) \approx$ ____________

Solution: We have that $(\nabla f)_{(1,1,1)} = (2, -1, 2)$ The linear approximation is $2 + (2, -1, 2) \cdot (0.2, 0.1, -0.1) = 2.1$.

(c) (5 points) Compute the curvature of $\mathbf{r}(t) = (6t, 3t^2, t^3)$ in $(0, 0, 0)$.
Answer: $\kappa(0, 0, 0) =$ ____________

Solution: We have $\kappa = |\mathbf{v} \times \mathbf{a}|/|\mathbf{v}|^3$, $\mathbf{v} = (6, 6t, 3t^2)$ and $\mathbf{a} = (0, 6, 6t)$. Thus, we seek

$$\frac{|(6, 0, 0) \times (0, 6, 0)|}{|(6, 0, 0)|^3} = \frac{6^2|(1, 0, 0) \times (0, 1, 0)|}{6^3} = \frac{6^2}{6^3} = \frac{1}{6}$$

3. (10 points) A surface is defined via $x^3 - xy + \arctan(xz) + 2z = 2$. Find the tangent plane in the point $(0, 1, 1)$.

$a)\ x + z = 1 \qquad b)\ 2x + 2z = 0 \qquad c)\ z = 1$

$d)\ 2x + 2z = 1 \qquad e)\ x + 2(z-1) = 0 \qquad f)\ 2x - 2z = 2$

Solution: Let $f(x, y, z) = x^3 - xy + \arctan(xz) + 2z - 2$. Then

$$\nabla f = (3x^2 - y + \frac{z}{1+(xz)^2}, -x, \frac{x}{1+(xz)^2} + 2),$$

and $(\nabla f)_{(0,1,1)} = (0, 0, 2)$. The equation for the tangent plane is therefore $2(z-1) = 0$ which can be expressed as $z = 1$.

4. (8 points) Let $c(t) = t^2$, and suppose the differentiable function $f(x, y)$ satisfies the following:

$$\lim_{h\to 0} \frac{f(c(1+h), 4) - f(c(1), 4)}{h} = 6, \qquad \lim_{h\to 0} \frac{f(1, c(2+h)) - f(1, c(2))}{h} = -12.$$

Find $(\nabla f)_P$ in the point $P = (1, 4)$. **Answer:** ____________

Solution: Note that the first limit is the definition of $\frac{d}{dt}(F(c(t), 4))$ at $t = 1$. According to the one-dimensional chain rule, the first limit therefore tells us that

$$F'_x(c(1), 4)c'(1) = 6 \qquad \Rightarrow \qquad F'_x(1, 4) \cdot 2 = 6$$

This implies that $F'_x(1, 4) = 3$. The same type of analysis shows that

$$F'_y(1, 4) \cdot c'(2) = -12$$

so $F'_y(1, 4) = -3$.

Hence, $(\nabla f)_{(1,4)} = (3, -3)$.

5. (12 points) Find the global *maximum* of $f(x,y) = \frac{xy-x+3}{1+x+y}$ in the triangle $x \geq 0$, $y \geq 0$, $x + y \leq 4$.

a) 3/2 *b)* 2 *c)* 5/2 *d)* 3 *e)* 7/2 *f)* 4

Solution: We find the partial derivatives,

$$f'_x = \frac{y^2-4}{(1+x+y)^2}, \qquad f'_y = \frac{(x-1)(3+x)}{(1+x+y)^2}.$$

The system $f'_x = f'_y = 0$ leads to $x \in \{1, -3\}$, $y = \pm 2$. The only point inside the region is $(x,y) = (1,2)$, with function value 1. It remains to study the function in the vertices and the three one-dimensional boundaries.

Case: $x = 0$*:* Define $h_1(t) = f(0,t)$, for $0 \leq t \leq 4$. We get $h_1(t) = 3/(1+t)$, which is decreasing on $[0,4]$, so there are no local extrema on this boundary.

Case: $y = 0$*:* Define $h_2(t) = f(t,0)$, for $0 \leq t \leq 4$. We get $h_2(t) = 4/(1+t) - 1$ which is also decreasing on $[0,4]$.

Case: $y = 4 - x$*:* Define $h_3(t) = f(t, 4-t)$, for $0 \leq t \leq 4$. We get $h_3(t) = (3 + 3t - t^2)/5$, and $h'_3(t) = (3-2t)/5$. Hence, we have a local maximum for $t = 3/2$ and $h_3(3/2) = 21/20$.

It remains to check the corners: $f(0,0) = 3$, $f(4,0) = -1/5$, $f(0,4) = 3/5$.

Thus, the global maximum 3, which we obtain in $(0,0)$.

6. (10 points) Suppose $f(x, y)$ is a differentiable with continuous second derivatives and that $f''_{xx}(x, y) + f''_{yy}(x, y) = 1$ everywhere. Compute and simplify

$$\frac{\partial^2}{\partial u^2} f(au + v, u - av) + \frac{\partial^2}{\partial v^2} f(au + v, u - av),$$

where a is a constant. **Answer:** ____________

Solution: We have, using the chain rule, that

$$\begin{aligned}\frac{\partial^2}{\partial u^2} f(au + v, u - av) &= \frac{\partial}{\partial u}\left(f'_x \cdot a + f'_y \cdot 1\right) \\ &= \left[f''_{xx} \cdot a + f''_{xy} \cdot 1\right] a + \left[f''_{xy} \cdot a + f''_{yy} \cdot 1\right]\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial^2}{\partial v^2} f(au + v, u - av) &= \frac{\partial}{\partial v}\left(f'_x \cdot 1 + f'_y \cdot (-a)\right) \\ &= \left[f''_{xx} \cdot 1 + f''_{xy} \cdot (-a)\right] + \left[f''_{xy} \cdot 1 + f''_{yy} \cdot (-a)\right](-a)\end{aligned}$$

The sum is therefor $(1 + a^2)(f''_{xx} + f''_{yy})$, which then simplifies to $1 + a^2$.

7. (a) (3 points) Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^{50}+x^{50}y^{50}}{\sqrt{x^{100}+y^{100}}} \qquad \text{does not exist.}$$

Solution: Along $x = 0$, the limit is 0. Along the line $y = 0$, the limit is 1. Hence, the limit does not exist.

(b) (7 points) Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^4+x^2y^2}{\sqrt{x^4+y^2}} \qquad \text{exists.}$$

Hint: You may want to use polar coordinates and the sandwich method.

Solution: Note that the expression is non-negative whenever it is defined. Polar coordinates gives

$$\lim_{r\to 0} \frac{r^4\cos^4(t)+r^4\cos^2(t)\sin^2(t)}{\sqrt{r^4\cos^4(t)+r^2\sin^2(t)}} = \lim_{r\to 0} \frac{r^2\cos^4(t)+r^2\cos^2(t)\sin^2(t)}{\sqrt{\cos^4(t)+\sin^2(t)/r^2}}$$

Since $\sqrt{\cos^4(t)+\sin^2(t)/r^2} \geq \sqrt{\cos^4(t)}$, we have that

$$0 \leq \lim_{r\to 0} \frac{r^2\cos^4(t)+r^2\cos^2(t)\sin^2(t)}{\sqrt{\cos^4(t)+\sin^2(t)/r^2}} \leq \lim_{r\to 0} \frac{r^2\cos^4(t)+r^2\cos^2(t)\sin^2(t)}{\sqrt{\cos^4(t)}}$$

The latter simplifies to

$$\lim_{r\to 0} r^2[\cos^2(t)+\sin^2(t)]$$

which has limit 0.

Alternative solution: Note that if $|x| \leq 1$, then $x^2y^2 \leq y^2$. Thus, near $(0,0)$, we have the inequality

$$\frac{x^4+x^2y^2}{\sqrt{x^4+y^2}} \leq \frac{x^4+y^2}{\sqrt{x^4+y^2}} = \sqrt{x^4+y^2}$$

The latter clearly goes to 0 as $(x,y) \to 0$.

Extra page