Math 114 **Name (Print):** ____________________

Midterm 2 Redemption **Recitation section:** ____________________

1. For these questions, you do not need to show your work — only provide an answer.

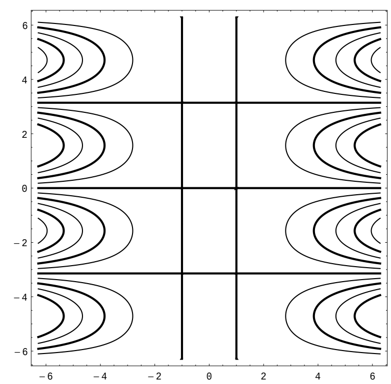
   (a) (4 points) Consider

$$f(x,y) = \begin{cases} 1 \text{ if } (x+y)(x-y)=0 \\ 0 \text{ otherwise.} \end{cases}$$

For which directions $\mathbf{u}$ does $(D_{\mathbf{u}}f)_{(0,0)}$ exist? **Underline all correct answers:**

$$(1,0), (-1,0), (0,1), (0,-1), \frac{(1,1)}{\sqrt{2}}, \frac{(-1,1)}{\sqrt{2}}, \frac{(1,-1)}{\sqrt{2}}, \frac{(-1,-1)}{\sqrt{2}}$$

   (b) (3 points) The level curves below correspond to which function?

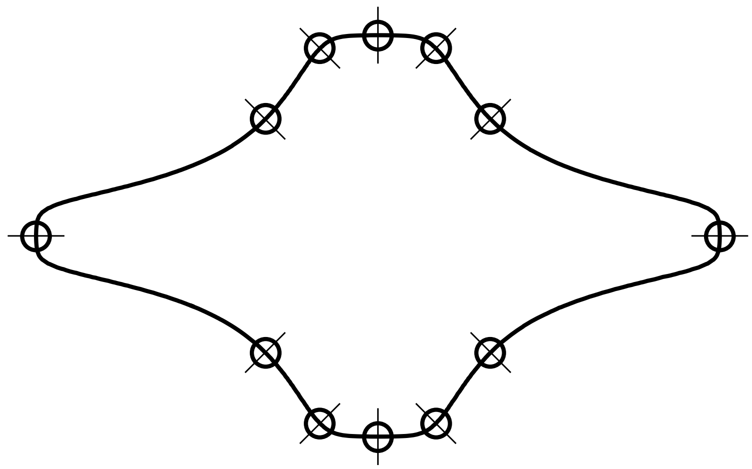


Underline one: $\cos(y)(x-1)$, $(x^2-1)\sin(y)$, $(y+1)\cos(y)$, $x\sin(y)$, $(x^2+1)\sin(2y)$.

Solution: The answer is $(x^2-1)\sin(y)$. It is the only one that vanish on $x=\pm 1$.

   (c) (3 points) The figure below shows the set $f(x,y)=0$ in the xy-plane for some complicated f. This defines $y(x)$ implicitly. For some points on the curve, we can compute ∇f, but not $y'(x)$.

Clearly mark these points in the picture.



Solution: The points with a vertical tangent cannot have a derivative $y'(x)$. These are the two points on the far left and far right.

2. For these questions, correct answer gives full score. Some points might be given for partial work.

(a) (3 points) Let $f(x, y)$ be differentiable, and suppose that in $(0, 0)$, the gradient is $(0, 0)$ and $f''_{xx} = 8$, $f''_{yy} = 2$ and $f''_{xy} = a$. For which a is this information not sufficient to characterize the critical point in $(0, 0)$?

Underline all correct answers: 1, −1, 4, −4, 0

> **Solution:** The formula tells us that the Hessian is $8 \cdot 2 - a^2 = 16 - a^2$. If $a = \pm 4$, the Hessian is 0, and we cannot characterize the critical point.

(b) (2 points) Suppose $\nabla f = (3z - yz, xz)$ and $f(1, 1) = 429$. Estimate $f(1.1, 1.2) - f(1, 1)$ using the standard linear approximation.

Answer: $f(1.1, 1.2) - f(1, 1) \approx$ ____________

> **Solution:** We have that $(\nabla f)_{(1,1)} = (2, 1)$ The difference is therefore approximately $(2, 1) \cdot (0.1, 0.2) = 0.4$.

(c) (3 points) Suppose that at time t_0, the total acceleration of a particle is $9m/s^2$ and we express the acceleration as $a = a_T\mathbf{T} + a_N\mathbf{N}$. Furthermore, the tangential component $a_T\mathbf{T}$ of the acceleration is $(1, 2, 0)$, and the principal normal vector $\mathbf{N}$ is $(0, 0, 1)$. Find the a_N at time t_0.

Answer: $a_N =$ ____________

> **Solution:** The formula for acceleration tells us that $9 = (a_T)^2 + (a_N)^2$. Now, $|a_T\mathbf{T}|^2 = |(1, 2, 0)|^2 = 1 + 4 = 5$. Hence, $a_T = \sqrt{5}$. Thus, $a_N^2 = 9 - 5$, so $a_N = 2$.

3. (10 points) Consider the function $f(x, y) = 1 + x^3 + xy + \arctan(xy)$. Find the tangent plane in the point $(0, 1)$.

$a)\ x + z = 1 \qquad b)\ 2x + 2z = 0 \qquad c)\ z = 1$

$d)\ 2x + 2z = 1 \qquad e)\ x + 2(z - 1) = 0 \qquad f)\ 2x - z + 1 = 0$

Solution: We have that $f(0, 1) = 1$,

$$\nabla f = (3x^2 + y + \frac{y}{1 + (xy)^2}, x + \frac{x}{1 + (xy)^2}),$$

and $(\nabla f)_{(0,1)} = (2, 0)$. The equation for the tangent plane is therefore $2x + 0(y - 1) - (z - 1) = 0$ which can be expressed as $2x - z + 1 = 0$.

4. (8 points) Let $f(x,y)$ be a differentiable function that satisfies the following:

$$\lim_{h\to 0} \frac{f(1+3h,2)-f(1,2)}{f(1,2+9h)-f(1,2)} = 4 \text{ and } f(1,2)=0.$$

The relation $f(x,y)=0$ defines y as a function of x implicitly. Find dy/dx in the point $(1,2)$. **Answer:** ____________

Solution: We rewrite the limit as

$$\lim_{h\to 0} \frac{f(1+3h,2)-f(1,2)}{f(1,2+9h)-f(1,2)} = \lim_{h\to 0} \frac{3}{9}\frac{f(1+3h,2)-f(1,2)}{3h} \cdot \frac{9h}{f(1,2+9h)-f(1,2)}$$

Hence,

$$\lim_{h\to 0} \frac{f(1+3h,2)-f(1,2)}{3h} \cdot \frac{9h}{f(1,2+9h)-f(1,2)} = 12$$

We recognize that

$$\lim_{h\to 0} \frac{f(1+3h,2)-f(1,2)}{3h} = f'_x(1,2) \text{ and } \lim_{h\to 0} \frac{f(1,2+9h)-f(1,2)}{9h} = f'_y(1,2)$$

so the limit must then be equal to f'_x/f'_y in $(1,2)$. Now, we know that $dy/dx = -f'_x/f'_y$, and it follows that dy/dx in $(1,2)$ is -12.

5. Let $f(x, y) = x(1 + y^2)$.
   (a) (2 points) Show that f has no points where ∇f is the zero vector.
   (b) (8 points) Find all three critical points of f on the curve $y^2 + x^3 = 1$ using Lagrange multipliers.
   (c) (4 points) Find the global *maximum* of f in the region bounded by $x \geq 0$ and $y^2 + x^3 \leq 1$.

a) $1/2$ *b)* $2^{1/3}$ *c)* $\frac{3}{2^{4/3}}$ *d)* 2 *e)* $2^{4/3}$ *f)* 3

Solution: Part a: We have that

$$\nabla f = (1 + y^2, 2xy),$$

and this is never the 0 vector since $1 + y^2$ can never be 0.

Solution: Part b: Let $g = y^2 + x^3 - 1$. Then $\nabla g = (3x^2, 2y)$ so we seek points when ∇f and ∇g are parallel, using Lagranges method. We get the system

$$1 + y^2 = \lambda \cdot 3x^2 \qquad 2xy = \lambda \cdot 2y.$$

The second equation gives either $\lambda = x$ or $y = 0$.

Case $\lambda = x$: the first equation gives $y^2 = 3x^3 - 1$. We plug that into $g(x, y) = 0$ and get

$$(3x^3 - 1) + x^3 - 1 = 0 \qquad \Rightarrow \qquad 4x^3 = 2 \qquad \Rightarrow \qquad x = 2^{-1/3}$$

Now $y^2 = 3x^3 - 1$ implies $y = \pm 2^{-1/2}$. Hence, we have the critical points $(2^{-1/3}, \pm 2^{-1/2})$.

Case $y = 0$: We plug that into $g(x, y) = 0$ and get $x^3 - 1 = 0$ so $x = 1$. We have the point $(1, 0)$ as another critical point.

Solution: Part c: Part a showed that there are no critical points inside the region, and part b found all critical points on one of the boundaries.

We must examine the boundary $x = 0$. On this line, f is constant and equal to 0.

Hence, the maximum is among $f(0, 0) = 0$, $f(1, 0) = 1$ and $f(2^{-1/3}, \pm 2^{-1/2}) = \frac{3}{2^{4/3}}$, where the latter is the maximum.

6. (10 points) Suppose $f(x,y)$ is a differentiable with continuous second derivatives and that $f''_{xx}(x,y) + f''_{yy}(x,y) = 1$ everywhere. Compute and simplify

$$\frac{\partial^2}{\partial u^2} f(2u+3v, 3u-2v) + \frac{\partial^2}{\partial v^2} f(2u+3v, 3u-2v)$$

Answer: ____________

Solution: The answer is $2^2+3^2 = 13$. See the other midterm 2 for complete solution.

7. (a) (4 points) Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{(xy)^2}{x^4 - y^4} \qquad \text{does not exist.}$$

Solution: Along $x = 0$, the limit is 0. Along the line $x = 2y$, we get

$$\lim_{y\to 0} \frac{(2y^2)^2}{16y^4 - y^4} = \lim_{y\to 0} \frac{4y^2}{15y^2} = 4/15$$

and thus the limit does not exist.

(b) (6 points) Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + xy^2}{e^{x^2+y^2} - 1} \qquad \text{exists.}$$

You may use the standard limit $\lim_{t\to 0} \frac{e^t - 1}{t} = 1$.

Solution: Polar coordinates and some rewriting gives

$$\lim_{r\to 0} r\cos(t)\frac{r^2}{e^{r^2} - 1} = \frac{\lim_{r\to 0} r\cos(t)}{\lim_{r\to 0} \frac{e^{r^2}-1}{r^2}} = \frac{0}{1} = 0$$

where we got the limit in the denominator using the fact that $\frac{e^r - 1}{r}$ tends to 1 as $r \to 0$.